A New Class of Non-stationary Interpolatory Subdivision Schemes based on Exponential Polynomials

Yoo-Joo Choi[1], Yeon-Ju Lee[2], Jungho Yoon[2], Byung-Gook Lee[3], and Young J. Kim[4,⋆]

[1] Dept. of CS., Seoul Univ. of Venture and Info., Seoul, Korea
yjchoi@suv.ac.kr
[2] Dept. of Math., Ewha Womans Univ., Seoul, Korea
{lee08,yoon}@ewha.ac.kr
[3] Div. of Internet Engineering, Dongseo Univ., Busan, Korea
lbg@dongseo.ac.kr
[4] Dept. of CS., Ewha Womans Univ., Seoul, Korea
kimy@ewha.ac.kr

Abstract. We present a new class of non-stationary, interpolatory subdivision schemes that can exactly reconstruct parametric surfaces including exponential polynomials. The subdivision rules in our scheme are interpolatory and are obtained using the property of reproducing exponential polynomials which constitute a *shift-invariant space*. It enables our scheme to exactly reproduce rotational features in surfaces which have trigonometric polynomials in their parametric equations. And the mask of our scheme converges to that of the polynomial-based scheme, so that the analytical smoothness of our scheme can be inferred from the smoothness of the polynomial based scheme.

1 Introduction

Subdivision surfaces are defined in terms of successive refinement rules that can generate smooth surfaces from initial coarse meshes. More formally, starting with the coarse control points $\mathbf{P^0} = \{\, \mathbf{p^0_n} \mid n \in Z^d \}$, recursive application of the subdivision rule S_k defines a new denser set of points $\mathbf{P^k} = \{\, \mathbf{p^k_n} \mid n \in Z^d \}$ which can be written as

$$\mathbf{P^k} = S_k \cdots S_0 \mathbf{P^0}, \quad k \in Z_+.$$

Here, a subdivision scheme is said to be stationary if S_k is the same regardless of k; otherwise it is called non-stationary [1]. Interpolatory subdivision schemes refine data by inserting values corresponding to intermediate points using a linear combination of neighboring points. As a result, in the limit, they keep the original data exactly. Furthermore, interpolatory subdivision has become a tool for multi-resolution analysis and wavelet construction on general manifolds (even in complicated geometric situations) using the *lifting scheme* [2].

Recently, concepts of refinement taken from signal processing have been applied to refinement of a triangular mesh in digital geometry processing. In the context of signal processing, a subdivision scheme can be seen as upsampling followed by filtering [3].

⋆ Corresponding author. TEL:+82-2-3277-4068

Moreover, in the univariate case it has been shown that a non-stationary subdivision scheme based on exponential or trigonometric polynomials with a suitable frequency factor works more effectively on highly oscillatory signals than stationary interpolatory scheme based on polynomials [4]. Thus, for instance, the Butterfly scheme [5], which is based on cubic polynomial interpolation, may not accurately reproduce highly oscillatory triangular mesh data. Motivated by these issues, we introduce a new class of non-stationary interpolatory subdivision schemes that can exactly reproduce a complicated, parametric surface including exponential polynomials in the sense of complex numbers; thus, our scheme can handle trigonometric and exponential functions as well as polynomials. The main idea of our scheme is that exponential polynomials constitute a *shift-invariant* space, and the mask of our subdivision scheme is constructed in such a way as to find the values of the exponential polynomials that correspond to the initial control points. As a result, the subdivision converges to the original surface as the subdivision level increases. Moreover, the shift-invariant property ensures that local weights corresponding to local control points are invariant, regardless of their locations, which ultimately enables parametric surfaces to be generated exactly. Furthermore, thanks to the linearity of our subdivision rules, a complicated surface generated from both polynomials and trigonometric polynomials can be exactly reproduced from a set of coarse initial points.

Not much work has been reported on the accurate reconstruction of given complicated surfaces by subdivision. Hoppe et al. [6] presented a method to reconstruct piecewise smooth surface models from scattered range data using a variation of Loop's scheme [7]. Morin et al. [8] addressed the issue of reconstructing rotational features in surfaces as special cases. Jena et al. [9] and Joe Warren [10] presented a non-interpolatory scheme based on the exponential splines for curve. But the exponential splines are non-interpolatory scheme for curves. Spline schemes such as [9] aim at reproducing only trigonometric curves like a circle, elliptic, helix but not surfaces. Jena et al. [11] presented a non-interpolatory scheme for tensor product bi-quadratic trigonometric spline surfaces.

2 Interpolatory Subdivision Scheme

2.1 Construction Rules

The construction of our subdivision scheme is based on the property of reproducing a certain class of exponential polynomials, which actually constitute a parametric surface F(u,v) defined on a planar parametric domain $\Omega \subset R^2$, where $F(u,v)$ can contain polynomials, trigonometric and exponential functions. Interpolatory subdivision schemes refine data by inserting new vertices, corresponding to intermediate points, into a planar triangulation by using linear combinations of neighboring vertices. The sampling density of the initial control points is assumed to satisfy the Nyquist rate [12]. The subdivision rules proposed in this paper are constructed so as to accommodate the class S of functions capable of producing parametric surfaces such as trigonometric surfaces and rotational features. In order to formulate a non-stationary rule, the space S is required to be *shift invariant* [13]: that is,

$$f \in S \quad \text{implies} \quad f(\cdot - \alpha) \in S, \quad \alpha \in Z. \tag{1}$$

Central ingredients in our construction are exponential polynomials of the form

$$\phi(u,v) = u^{\alpha_1} v^{\alpha_2} e^{\beta_1 u} e^{\beta_2 v}, \quad (u,v) \in \Omega \subset R^2, \tag{2}$$

where $\alpha_i = 0, \cdots, \mu$ for some non-negative integer μ and complex numbers β_i $(i = 1, 2)$.

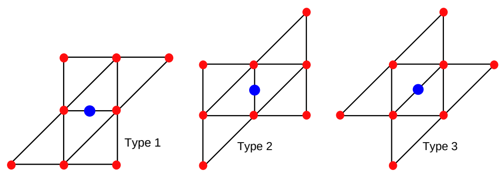

Fig. 1. *Stencils based on butterfly shapes. The blue dot represents the new insertion point.*

Let

$$S = \mathrm{span}\{\phi_n(u,v) | n = 1, \cdots, N\}$$

be a shift-invariant space with linearly independent ϕ_n's of the form given in Eq. 2. For each level (say k), the non-stationary subdivision rule is constructed by solving the linear system:

$$\phi_n(p2^{-k-1}) = \sum_{\ell \in \mathscr{X}} a^{[k]}_{p-2\ell} \phi_n(\ell 2^{-k}), \quad \phi_n \in S, \ k = 0, 1, \cdots, \tag{3}$$

where $p/2$ is the insertion point in the triangulation at level 0, and $\mathscr{X}$ is its corresponding stencil (see Fig.1). This linear system can be written in matrix form as

$$\mathbf{a}^{[k]} = \mathbf{B}^{[k]^{-1}} \mathbf{b}^{[k]}, \tag{4}$$

where $\mathbf{a}^{[k]} = (a^{[k]}_{p-2\ell} : \ell \in \mathscr{X})$, $\mathbf{B}^{[k]} = (\phi_n(\ell 2^{-k}) : \ell \in \mathscr{X}, \ n = 1, \cdots, N)$ and $\mathbf{b}^{[k]} = (\phi_n(p2^{-k-1}) : n = 1, \cdots, N)$. Refining the triangular mesh involves three groups of vertices (termed stencils) to evaluate three types of new vertices depending on their locations, as shown in Fig.1. There are then two important arguments to be discussed:

– **Uniqueness:** For each stencil (say $\mathscr{X}$) of the rule it is required that $\dim(S|_{\mathscr{X}}) = \dim S$. This guarantees a unique solution to the linear system (Eq. 3).

– **Non-stationary rule:** The shift-invariant property implies that the rule is the same everywhere at the same level k but may vary between different levels.

The configuration of the stencil $\mathscr{X}$ and the space S may differ, depending on the target surface $F(u,v)$. We begin with a butterfly-shaped stencil (shown in Fig.1) which can be considered as a non-stationary version of the well-known Butterfly scheme [5]. We will see that this new scheme provides the same smoothness and approximation order as the original Butterfly scheme, with the additional advantages that: (1) it reproduces certain types of rotational features; (2) there are flexibilities in the choice of S and frequency factors β_i $(i = 1, 2)$ in Eq. 2. In fact, any surface whose parametric equations constitute a shift-invariant space with fewer than eight basis functions can be reconstructed exactly using this butterfly stencil. As simple examples we take the sphere and torus.

Example 1. (Sphere and Torus) Let us define S as:

$$S := \text{span}\{\ \sin u, \cos u, \sin v, \cos v,\ \sin u \sin v, \cos u \cos v, \sin u \cos v, \cos u \sin v\}.$$

Then the three types of refinement rule are obtained by solving the linear system of Eq. 3 using the butterfly stencil. Recursive application of this subdivision rule exactly generates a sphere whose parametric equation $F(u,v)$, $0 \le v \le 2\pi,\ 0 \le u \le \pi$, is:

$$x(u,v) = r \sin u \cos v, \quad y(u,v) = r \sin u \sin v, \quad z(u,v) = r \cos u.$$

Similarly, the non-stationary scheme reconstructs a torus whose $F(u,v)$ is given by

$$x(u,v) = (1+\cos u)\cos v,\ y(u,v) = (1+\cos u)\sin v,\ z(u,v) = \sin u,$$

where $0 \le u, v \le 2\pi$.

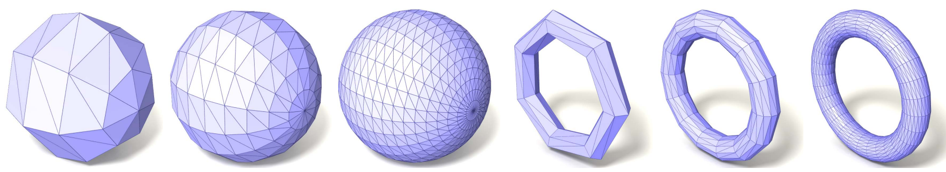

Fig. 2. *Exact reconstruction of a sphere and a torus.*

Later we will introduce additional types of stencil which can reproduce more complicated parametric surfaces.

2.2 Asymptotic Equivalence

As the level of refinement increases, the mask at level k, $\{a_n^{[k]}\}$, of our non-stationary scheme converges to that, $\{a_n\}$, of a stationary scheme based on polynomial interpolation; this property is also known as 'asymptotic equivalence' between the two schemes in the sense of

$$|a_n^{[k]} - a_n| = O(2^{-k})$$

as k increases [14]. The asymptotic equivalence property guarantees that, in the limit, the non-stationary subdivision converges uniformly to a continuous surface. This observation plays a key role in proving that the non-stationary scheme has the same smoothness as the stationary Butterfly scheme.

2.3 Smoothness and Approximation Order

The central ingredient in the analysis of non-stationary schemes is the asymptotic equivalent relation between schemes as shown in Thm 1. The analytical properties of a stationary scheme are well-understood [15], and so the smoothness of a non-stationary scheme can be inferred from the stationary scheme to which it is asymptotically equivalent. On the other hand, another important issue in devising subdivision algorithms is how to attain the original function as closely as possible when the initial data is sampled from the underlying function. A high quality reconstruction scheme should guarantee that the approximation error decreases as the sample rates increase. We can find the following results from [14]:

Theorem 1. *Let $\{S_{a^{[k]}}\}$ is the non-stationary interpolatory subdivision scheme with a butterfly stencil. Then, we have*

1. *The scheme $\{S_{a^{[k]}}\}$ is C^1, i.e., it has the same smoothness as the stationary Butterfly subdivision scheme $\{S_a\}$.*
2. *The scheme $\{S_{a^{[k]}}\}$ has the approximation order 4 on any compact K in R^2.*

3 Reconstruction of Mathematical Surfaces

In this section, we extend the basic interpolatory subdivision scheme to a more general scheme which is able to reconstruct mathematical parametric surfaces. For a given parametric surface F, we first explain how to construct a shift-invariant space S along with its basis functions that constitute F, and then provide a general algorithm that creates a stencil $\mathscr{X}$ upon which our subdivision scheme to reconstruct F is based.

3.1 Finding the Shift Invariant Space

From a given parametric surface F, our goal is to find its shift invariant space, S, in the smallest dimension: we want to minimize the dimension of S since we want to use as few vertices as possible in the subdivision rules.

To construct the shift-invariant space S, we first search for a finite collection of functions, B, whose elements generate F via a linear combination. To find B, we initialize B as an empty set and enumerate all the linearly independent monomials constituting F. Then we incrementally add each of these monomials to B, provided that the monomial is linearly independent to the elements already added to B. Once we have B, we find S that generates S as follows ;

1. Initially, we set S to be the same as span B.
2. We pick an element f_i from S and perform a constant shifting of f_i as shown in Eq. 1; i.e., compute $f_i(\cdot - \alpha)$.
3. We enumerate all the monomials f_i^j constituting $f_i(\cdot - \alpha)$.
4. For each f_i^j, if it can be generated by the current S, then we do not include it in S and continue to check other f_i^js; otherwise we add it to S.
5. Repeat between 2 and 4 until there is no possible new addition to S.
6. Finally, we have the shift invariant space $S := \operatorname{span} S$.

3.2 Stencil Generation Algorithm

Once we have formed the shift-invariant space S for a given surface F, and an appropriate stencil $\mathscr{X}$, then the linear system in Eq. 4 must have a unique solution. This means that the stencil $\mathscr{X}$ must satisfy $\dim(S|_{\mathscr{X}}) = \dim S$ and, equivalently, $\dim \mathbf{B} = \dim S$ in Eq. 4; i.e., $\mathbf{B}$ is invertible. The solution of this system provides a mask set corresponding to the stencil $\mathscr{X}$ in the subdivision rule. However, the stencil that satisfies the linear system in Eq. 3 is not necessarily unique. Moreover, we need to keep the stencil symmetric and concentrated around the vertex to be refined, so that the resulting mask set is symmetric and has as small a support as possible. This property preserves the locality of subdivision rules and reduces the computational costs when these rules are applied.

We construct a stencil $\mathscr{X}$ starting from a newly inserted point p. First, we choose two stencil vertices (say v_1 and v_2) connected to p and continue to search for other stencil vertices by expanding v_1 and v_2 to their neighborhood (more specifically, n-ring neighbors) while keeping the stencil shape symmetric. Here, the n-ring neighbors of v are defined as vertices that are reachable from v by traversing no more than n edges in the mesh. This search process continues until the associated matrix $\mathbf{B}$ satisfies $\dim \mathbf{B} = \dim S$. This search process can be efficiently implemented using breadth first traversal of the initial subdivision mesh.

We will now provide an example of generating stencils for a complicated surface: the Möbius strip.

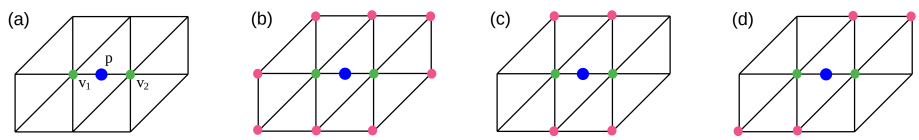

Fig. 3. *Type 1 stencil for a Möbius strip. (a) The newly inserted point p is colored blue and the two green points are chosen as the immediate neighbors (v_1, v_2). (b) The one-ring neighbors of v_1, v_2 are colored pink. (c) A candidate stencil, turns out to be invalid. (d) A valid stencil.*

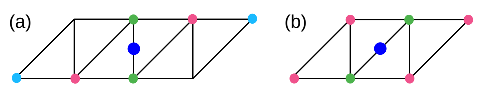

Fig. 4. *Type 2 and 3 stencils for a Möbius strip.*

Example 2. (Möbius strip) The parametric equation $F(u,v)$ for Möbius strip is given by

$$\begin{aligned} x(u,v) &= a\cos u + v\cos(u/2), \\ y(u,v) &= a\sin u + v\cos(u/2), \\ z(u,v) &= v\sin(u/2), \end{aligned}$$

where $0 \le u \le 2\pi$, $-w \le v \le w$, and w and a are constants.

Then the space that generates the parametric surface with the smallest dimension is

$$B := \{\sin u, \cos u, v\sin(u/2), v\cos(u/2)\},$$

and the corresponding shift-invariant space S that generates $F(u,v)$ is

$$S := \text{span}\{\sin u, \cos u, \sin(u/2), \cos(u/2), v\sin(u/2), v\cos(u/2)\}.$$

To find the stencil, we initiate a search from the newly inserted point p as shown in Fig. 3-(a) (Type 1 stencil). Then, we choose two closest vertices (the green dots in Fig. 3-(a)), v_1, v_2, connected to p. Since $\dim S = 6$, we now need to find four more vertices. We may choose any vertex among the one-ring neighbors (pink dots in Fig. 3-(b)) of $\{v_1, v_2\}$; however, to preserve the symmetry, we may choose four vertices like those in Fig. 3-(c). But the resulting stencil does not satisfy $\dim \mathbf{B} = 6$, so we discard these vertices and seek others. Finally, we locate the four vertices shown in Fig. 3-(d); since the resulting stencil satisfies $\dim \mathbf{B} = 6$, we can stop the search. We can find stencils for different types of points in a similar way, as shown in Fig. 4-(a) (Type 2) and 4-(b) (Type 3). Notice that in case of Type 2 point, we include two-ring neighbors (light blue dots) to create a symmetric stencil, because $\dim \mathbf{B} = 6$ cannot be satisfied with one-ring neighbors alone.

Fig. 5 shows the reconstruction results of two benchmarking models. A figure-8 klein bottle is defined by

$$\begin{aligned} x(u,v) &= (a + \cos(u/2)\sin v - sin(u/2)\sin(2v))\cos u, \\ y(u,v) &= (a + \cos(u/2)\sin v - sin(u/2)\sin(2v))\sin u, \\ z(u,v) &= \sin(u/2)\sin v + \cos(u/2)\sin(2v), \end{aligned}$$

where $0 \le u \le 2\pi$, $0 \le v \le 2\pi$ and some constant a.

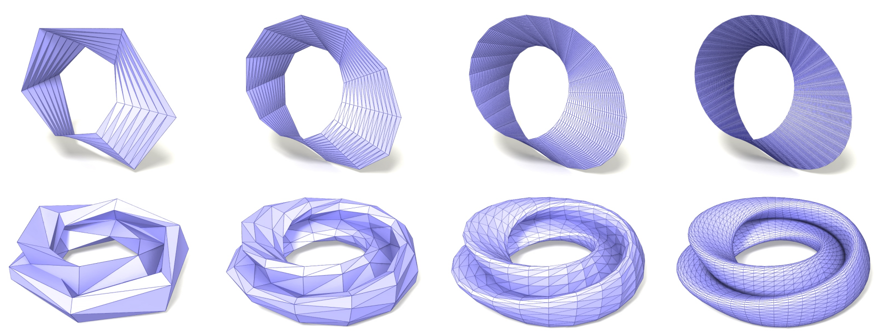

Fig. 5. *Reconstruction of complicated surfaces. The rows, from top to bottom, show the reconstruction of a möbius strip and a figure-8 klein bottle. The images, from left to right, show the initial mesh, and subdivision levels 1, 2, and 3.*

4 Limitations and Future Work

Our scheme cannot reconstruct parametric surfaces that contain non-exponential polynomials such as a logarithmic functions and division terms, which actually require non-uniform masks of subdivision for the exact reconstruction of such functions. Nor can it handle a mesh with arbitrary topology. There are several areas for future work. First of all, extending our scheme to a mesh with arbitrary topology is an immediate challenge. We would also like to work on regenerating exact surface normals by subdivision. And if we could reconstruct surfaces containing singular points we would be able to address many additional interesting applications.

Acknowledgements

Young J. Kim is sponsored in part by the grant KRF-2004-205-D00168 of the KRF funded by the Korean government, the ITRC program and the MOST STAR program. Jungho Yoon and Yeon-Ju Lee are supported in part by the grant KRF-2005-015-C00015 funded by Korea Government (MOEHRD, Basic Reasearch Promotion Fund).

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